UNIVERSITY COLLEGE LONDON

EXAMINATION FOR INTERNAL STUDENTS

MODULE CODE : **MATH1402**

ASSESSMENT PATTERN : **MATH1402A**

MODULE NAME : **Mathematical Methods 2**

DATE : **28-Apr-08**

TIME : **10:00**

TIME ALLOWED : **2 Hours 0 Minutes**

2007/08-MATH1402A-001-EXAM-204

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. (a) State, without proof, the general formula for a Fourier series on $(-\pi, \pi)$ for a function $f(x)$, giving the expressions for the coefficients.

   (b) Find the Fourier series for

$$f(x) = \begin{cases} 0 & , \quad \text{if } -\pi < x \leqslant -\frac{\pi}{2} \\ 1 & , \quad \text{if } -\frac{\pi}{2} < x \leqslant \frac{\pi}{2} \\ 0 & , \quad \text{if } \frac{\pi}{2} < x < \pi \end{cases} .$$

   (c) Using Part b, or otherwise, show that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots .$$

2. (a) Define the Jacobian

$$\frac{\partial(x, y)}{\partial(u, v)},$$

   where $x(u, v)$ and $y(u, v)$ are smooth functions.

   (b) Using the definition from Part a, determine the Jacobian for the coordinate transformation defined by the functions

$$x(u, v) = \frac{u + v}{2} \quad \text{and} \quad y(u, v) = \frac{u - v}{2}.$$

   (c) Let R be the first quadrant in the xy-plane. Using the coordinate transformation from Part b, or otherwise, find

$$\iint_R e^{-(x+y)^2}\, dx dy.$$

3. (a) State the Divergence Theorem carefully.

(b) Let S be the surface of the closed box defined by $0 \leqslant x \leqslant 4$, $0 \leqslant y \leqslant 2$, and $0 \leqslant z \leqslant 10$. Let a be a positive constant and consider the vector field

$$\boldsymbol{F}(x,y,z) = (3a^3 - 2a)x\boldsymbol{i} + \frac{x^2\cos(z)}{z+4}\boldsymbol{j} + \frac{\ln(|y+1|)}{(x+1)^2}\boldsymbol{k}.$$

Find the exact value of a so that the flux of $\boldsymbol{F}$ over S is 0.

(c) Considering the situation in Part b, how would the value of a change if we replace the vector field $\boldsymbol{F}$ by the vector field $\boldsymbol{F} + \boldsymbol{G}$, where $\boldsymbol{G}(x,y,z) = g_1(y,z)\boldsymbol{i} + g_2(x,z)\boldsymbol{j} + g_3(x,y)\boldsymbol{k}$ with smooth functions $g_1, g_2, g_3 : \mathbb{R}^2 \longrightarrow \mathbb{R}$.

4. (a) State Green's Theorem in the plane carefully.

(b) Sketch the contour C which is described as follows: Begin at the point $(2,3)$. Go to the point $(-2,3)$ along the straight line segment. Then go back to $(2,3)$ along the curve given by the equation $y = x^2 - 1$. This description also gives you the correct orientation of C.

(c) Let $\boldsymbol{F}(x,y) = x\sin(x)\boldsymbol{i} + (xy + \ln(1+y^2))\boldsymbol{j}$. Use Green's Theorem to calculate the circulation of $\boldsymbol{F}$ around C.

5. (a) State Stoke's Theorem carefully.

(b) Verify Stoke's Theorem for the vector field

$$\boldsymbol{F}(x,y,z) = -y\boldsymbol{i} + x\boldsymbol{j} + xz\boldsymbol{k}$$

and the surface S defined by $x^2 + y^2 + z^2 = 17$ and $z \geqslant 4$. Sketch the surface S.

6. (a) Let $\boldsymbol{A}$ be a vector potential for $\boldsymbol{B}$, i.e., $\boldsymbol{B} = \text{curl}\boldsymbol{A}$. Let $\phi : \mathbb{R}^3 \longrightarrow \mathbb{R}$ be a smooth function and show that

$$\boldsymbol{A} + \vec{\nabla}\phi$$

is also a vector potential for $\boldsymbol{B}$. Find an expression for the divergence of $\boldsymbol{A}+\vec{\nabla}\phi$ in terms of the divergence of $\boldsymbol{A}$.

(b) For the vector potential $\boldsymbol{A} = 2x\boldsymbol{i} + 2y\boldsymbol{j} + 2z\boldsymbol{k}$, is it possible to find a smooth function $\phi : \mathbb{R}^3 \longrightarrow \mathbb{R}$ such that $\boldsymbol{A} + \vec{\nabla}\phi$ is divergence free? If so, provide a ϕ that works.

(c) A central vector field is one of the form $\boldsymbol{F} = f(r)\boldsymbol{r}$, where $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a smooth function, $\boldsymbol{r} + x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$, and $r = |\boldsymbol{r}|$. Show that any central vector field is irrotational, i.e., $\text{curl}\boldsymbol{F} = 0$.